COHOMOLOGY OF FUCHSIAN GROUPS AND FOURIER INTERPOLATION

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Abstract. We give a new proof of a Fourier interpolation result first proved by Radchenko-Viazovska, deriving it from a vanishing result of the first cohomology of a Fuchsian group with coefficients in the Weil representation.

Contents

1. Introduction

Let $\mathcal{S}$ be the space of even Schwartz functions on the real line, and $\mathfrak{s}$ the space of sequences of complex numbers $(a_n)_{n\geq 0}$ such that $|a_n|n^k$ is bounded for all k; we write $\hat{\phi}(k) = \int_{\mathbf{R}} \phi(x)e^{2\pi ikx}dx$ for the Fourier transform of $\phi \in \mathcal{S}$. In [RV19], Radchenko-Viazovska proved the following beautiful "interpolation formula":

Theorem 1.1. *The map*

$$\Psi : \mathcal{S} \to \mathfrak{s} \oplus \mathfrak{s}, \quad \phi \mapsto (\phi(\sqrt{n}), \hat{\phi}(\sqrt{n}))_{n\geq 0}$$

is an isomorphism onto the codimension 1 subspace of $\mathfrak{s} \oplus \mathfrak{s}$ cut out by the Poisson summation formula, i.e. the subspace of (x_n, y_n) defined by $\sum_{n\in\mathbf{Z}} x_{n^2} = \sum_{n\in\mathbf{Z}} y_{n^2}$.

The morphism Ψ is in fact a homeomorphism of topological vector spaces with reference to natural topologies. We will give another proof of this theorem. The first step of this proof is to notice that the evaluation points $\sqrt{n}$ occur very naturally in the theory of the *oscillator representation* defined by Segal–Shale–Weil. Using this observation, the theorem can be reduced to computing the cohomology of a certain Fuchsian group with coefficients in this oscillator representation, and here we prove a more general statement:

Theorem 1.2. *Let G be $\mathrm{SL}_2(\mathbf{R})$ or a finite cover, Γ a lattice in G, W an irreducible infinite-dimensional $(\mathfrak{g}, K)$-module, and $W^*_{-\infty}$ the distributional globalization of its dual (see (2.4)). Then $H^1(\Gamma, W^*_{-\infty})$ is always finite dimensional, and in fact*

$$\dim H^1(\Gamma, W^*_{-\infty}) = \textit{multiplicity of } W^{cl} \textit{ in cusp forms,} \tag{1}$$

where W^{cl} is the complementary irreducible representation to W defined in §2.3.

The theorem can be contrasted with usual Frobenius reciprocity:

(2) $\quad \dim H^0(\Gamma, W^*_{-\infty})$ = multiplicity of W in the space of automorphic forms,

Note that, in the passage from (1) to (2), "cusp forms" have been replaced by "automorphic forms" and W^{cl} by W. We also emphasize the surprising fact that, in the theorem, the H^1 takes *no account* of the topology on $W^*_{-\infty}$: it is simply the usual cohomology of the discrete group Γ acting on the abstract vector space $W^*_{-\infty}$. The corresponding determination for finite-dimensional W is the subject of automorphic cohomology and is in particular completely understood, going back to [Eic57].

A variant of Theorem 1.2, computing *all* the cohomology groups H^i when W is a spherical principal series representation, was already proved by Bunke and Olbrich in the 1990s. We were unaware of this work when we first proved Theorem 1.2; our original argument has many points in common with [BO98], most importantly in our usage of surjectivity of the Laplacian both for analytic and algebraic purposes, but also has some substantial differences of setup and emphasis. We will correspondingly give two proofs: the first based on the results of [BO98], and the second a shortened version of our original argument.

Some other interpolation consequences of Theorem 1.2 arise by replacing $\mathcal{S}$ by other spaces of functions carrying natural representations: we discuss this in §6.4. For example any function $f \in C^\infty(\mathbf{R})$ for which $g(x) := f(1/x)$ extends to a smooth function at 0 is determined by $\hat{f}(n), \hat{g}(n)$ for nonzero $n \in \frac{1}{2}\mathbb{Z}$, together with the values and first derivatives of f, g at 0.

1.1. Theorem 1.2 implies Theorem 1.1.

We give an outline of the argument and refer to §6 for details.

We pass first to a dual situation. Denote by $\mathcal{S}^*$ the space of tempered distributions, i.e. the continuous dual of $\mathcal{S}$. For our purposes we regard it as a vector space without topology.

Similarly, we define $\mathfrak{s}^*$ as the continuous dual of $\mathfrak{s}$, where $\mathfrak{s}$ is topologized by means of the norms $\|(b_n)\|_k := \sup_n b_n(1+|n|)^k$; thus, $\mathfrak{s}^*$ may be identified with sequences (a_n) of complex numbers of polynomial growth, where the pairing of $(a_n) \in \mathfrak{s}^*$ and $(b_n) \in \mathfrak{s}$ is given by the rule $\sum a_n b_n$. With this notation, the map

$$\Psi^* : \mathfrak{s}^* \oplus \mathfrak{s}^* \to \mathcal{S}^*,$$

dual to Ψ takes the coordinate functions to the distributions δ_n, and $\hat{\delta}_n$,

$$(a_n, b_n)_{n\geq 0} \mapsto \sum a_n \delta_n + b_n \hat{\delta}_n,$$

where

$$\delta_n(\phi) = \phi(\sqrt{n}), \quad \hat{\delta}_n(\phi) = \hat{\phi}(\sqrt{n}).$$

Then Theorem 1.1 is equivalent to the assertion:

> (Dual interpolation theorem): Ψ^* is surjective and its kernel consists precisely of the "Poisson summation" relation.

The equivalence of this statement and Theorem 1.1 is not a complete formality because of issues of topology: see (47) for an argument that uses a theorem of Banach.

The next key observation is that the space of distributions spanned by δ_n and by $\hat{\delta}_n$ occur in a natural way in representation theory.

> The closure of the span of δ_n (respectively, the closure of the span of $\hat{\delta}_n$) coincide with the e-fixed and f-fixed vectors on the space $\mathcal{S}^*$ of tempered distributions, where
>
> $$e = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad f = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \tag{3}$$
>
> act on $\mathcal{S}^*$ according to the oscillator representation (see §6.1 for details), namely by multiplication by $e^{2\pi i x^2}$ and its Fourier transform respectively.

Let Γ be the group generated by e and f inside $\mathrm{SL}_2(\mathbf{R})$: it is a free group, of index two in $\Gamma(2)$, and it lifts to G (which we take in this case to be the double cover of $\mathrm{SL}_2(\mathbf{R})$). As explicated in §6, computations of dimensions of modular forms and Theorem 1.2 yield

$$\dim H^0(\Gamma,\mathcal{S}^*) = 1, \quad \dim H^1(\Gamma,\mathcal{S}^*) = 0. \tag{4}$$

The final observation is that

> The kernel and cokernel of $(\mathcal{S}^*)^e \oplus (\mathcal{S}^*)^f \to \mathcal{S}^*$ compute, respectively, the H^0 and H^1 of Γ acting on $\mathcal{S}^*$.

This follows from a Mayer-Vietoris type long exact sequence that computes the cohomology of the free group Γ, namely,

$$0 \to H^0(\Gamma,\mathcal{S}^*) \to H^0(\langle e\rangle,\mathcal{S}^*) \oplus H^0(\langle f\rangle,\mathcal{S}^*) \to H^0(1,\mathcal{S}^*) \to H^1(\Gamma,\mathcal{S}^*) \to \cdots \tag{5}$$

Combined with (4), we see that $\mathcal{S}^* = (\mathcal{S}^*)^e + (\mathcal{S}^*)^f$, i.e. the desired surjectivity of Ψ^*, and that the intersection of $(\mathcal{S}^*)^e$ and $(\mathcal{S}^*)^f$ is one-dimensional; this corresponds exactly to the Poisson summation formula.

Another way to look at this is the following. The Poisson summation formula is an obstruction to *surjectivity* in Theorem 1.1 and is closely related to the invariance of the distribution $\sum \delta_n \in \mathcal{S}^*$ by Γ, i.e. the existence of a class in the *zeroth* cohomology of Γ on $\mathcal{S}^*$. The above discussion shows a less obvious statement: the obstruction to *injectivity* in Theorem 1.1 is precisely the *first* cohomology of Γ on $\mathcal{S}^*$.

1.2. **The proof of Theorem 1.2.** The analogue of Theorem 1.2 when W is finite-dimensional and $\Gamma\backslash G$ is compact is (by now) a straightforward exercise; as noted, the ideas go back at least to [Eic57], and the general case is documented in [BW00]; the noncompact case is less standard but also well-known, see e.g. [Cas84] and [Fra98] for a comprehensive treatment.

The main complication of our case is that the coefficients are infinite-dimensional and one might think this renders the question unmanageable. The key point is that W is irreducible as a G-module. This says that, "relative to G", it is just as good as a finite-dimensional representation.

We will present two proofs of Theorem 1.2:

- The first proof, in Section 3, relies on the work of Bunke-Olbrich who computed the cohomology of lattices in $\mathrm{SL}_2(\mathbf{R})$ with coefficients in (the distribution globalization of a) principal series representation. We summarize a sketch of the argument of [BO98] for the convenience of the reader, and also because their argument as written does not cover the situation we need.

To deduce Theorem 1.2 from these results then requires us to pass from a principal series to a subquotient, which we do in a rather *ad hoc* way.

- The second proof is our original argument prior to learning of the work of Bunke-Olbrich just mentioned. However, given the content of [BO98], we have permitted ourselves to abridge the most tedious parts of our original argument, and reproduce here in detail the part that is perhaps most distinct from [BO98] – namely, we express the desired cohomology groups in terms of certain Ext-groups of $(\mathfrak{g}, K)$-modules and then compute these explicitly. Although we do not give full details here, this proof is quite explicit, and in particular it should be possible to produce an interpolation basis by explicating every step.

In both arguments the surjectivity of a Laplacian type operator plays an essential role. Such results are known since the work of Casselman [Cas84], and in their work, Bunke–Olbrich prove and utilize such a result both at the level of G and $\Gamma\backslash G$. We include a self-contained proof of such a result for $\Gamma\backslash G$ in §5.

1.3. **Questions.** It is very interesting to ask about the situation where Γ is not a lattice. Indeed, if one were to ask about an interpolation formula with evaluation points $1.1\sqrt{n}$, one is immediately led to similar questions for a discrete but *infinite covolume* subgroup of $\mathrm{SL}_2(\mathbf{R})$. Kulikov, Nazarov and Sodin [KNS23] have recently shown very general results about Fourier uniqueness that imply, in particular, that evaluating f and $\hat{f}$ at $1.1\sqrt{n}$ do not suffice to determine f; it would be interesting to see if one can extract more precise information here by using the nonabelian harmonic analysis of the current paper.

Perhaps a more straightforward question is to an establish an isomorphism

$$H^i(\Gamma, W^*_{-\infty}) \simeq \mathrm{Ext}^i_{\mathfrak{g},K}(W, \text{space of automorphic forms for } \Gamma\backslash G), \tag{6}$$

which is valid for general lattices Γ in semisimple Lie groups G and general irreducible (smooth, moderate growth) representations V of G. Bunke and Olbrich have proved this in the cocompact case and our original argument proceeded by establishing the case $i = 1$ for general lattices in $\mathrm{SL}_2(\mathbf{R})$.

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2. Covering groups of $\mathrm{SL}_2(\mathbf{R})$

Let $q \geq 1$ be a positive integer and let G be the q-fold covering of the group $\mathrm{SL}_2(\mathbf{R})$, i.e. G is a connected Lie group equipped with a continuous homomorphism $G \to \mathrm{SL}_2(\mathbf{R})$ with kernel of order q. This characterizes G up to unique isomorphism covering the identity of $\mathrm{SL}_2(\mathbf{R})$.

Denote by $\mathfrak{g}$ the shared Lie algebra of G and of $SL_2(\mathbf{R})$ and $\exp : \mathfrak{g} \to G$ the exponential map. Also denote by K the preimage of $\mathrm{SO}_2(\mathbf{R})$ inside G; it is abstractly isomorphic as topological group to $S^1 = \mathbf{R}/\mathbf{Z}$ and we fix such an isomorphism below.

The quotient G/K is identified with the hyperbolic plane $\mathbf{H}$, on which G acts by isometries. Define the norm of $g \in G$ to be $\|g\| := e^{\mathrm{dist}_{\mathbf{H}}(i,gi)}$. Equivalently, we could use $\left\|\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\right\| = \sqrt{a^2+b^2+c^2+d^2}$ since either of these two norms is bounded by a constant multiple of the other.

2.1. **Lie algebra.** Let H, X, Y be the standard basis for $\mathfrak{g}$:

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We also use $\kappa = i(X-Y)$, $2p = H - i(X+Y)$, $2m = H + i(X+Y)$, or, in matrix form:

$$\kappa = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad 2p = \begin{pmatrix} 1 & -i \\ -i & -1 \end{pmatrix}, \quad 2m = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}. \tag{7}$$

We have $\kappa = ik$, where k generates the Lie algebra of K. The isomorphism $K \simeq \mathbf{R}/\mathbf{Z}$ will be normalized to take the element 1 in the Lie algebra of $\mathbf{R}/\mathbf{Z}$ to the element k.

The elements p, m and κ satisfy the commutation relations

$$[p,m] = \kappa, \quad [\kappa, p] = 2p, \quad [\kappa, m] = -2m, \tag{8}$$

which say that p and m raise and lower κ-weights by 2. The Casimir element $\mathcal{C}$ in the universal enveloping algebra determined by the trace form is given by any of the equivalent formulas:

$$\mathcal{C} = \frac{H^2}{2} + XY + YX = \frac{\kappa^2}{2} + pm + mp = \frac{\kappa^2}{2} + \kappa + 2mp = \frac{\kappa^2}{2} - \kappa + 2pm. \tag{9}$$

2.2. **Iwasawa decomposition.** There is a decomposition

$$G = NAK \tag{10}$$

where A and N are the connected Lie subgroups of G with Lie algebra $\mathbf{R}.H$ and $\mathbf{R}.X$ respectively. We will parameterize elements of A via $a(y) := \exp(\frac{1}{2}\log(y)H)$, so that $a(y)$ projects to the diagonal element of $\mathrm{SL}_2(\mathbf{R})$ with entries $y^{\pm 1/2}$. We will also write $n_x = \exp(xX)$.

2.3. $(\mathfrak{g}, K)$-**modules.** We recall that a $(\mathfrak{g}, K)$ module means a $\mathfrak{g}$-module equipped with a compatible continuous action of K. Equivalently, it is described by the following data:

- For each $\zeta \in q^{-1}\mathbf{Z}$, a vector space V_ζ giving the ζ-weight space of K, so that κ acts on V_ζ by ζ;
- maps $p : V_\zeta \to V_{\zeta+2}$ and $m : V_\zeta \to V_{\zeta-2}$ satisfying $[p,m] = \kappa$.

We recall some facts about classification, see [HT12] for details. Irreducible, infinite-dimensional $(\mathfrak{g}, K)$-modules belong to one of three classes; in each case, the weight spaces V_ζ have dimension either zero or 1.

- Highest weight modules of weight ζ; these are determined up to isomorphism by the fact that their nonzero weight spaces occur in weights $\{\zeta, \zeta-2, \zeta-4, \dots\}$. V_ζ is killed by p. One computes using (9) that on such modules, the Casimir element $\mathcal{C}$ acts by $\zeta(\zeta+2)/2$.
- Lowest weight modules of weight ζ; these are determined up to isomorphism by the fact that their nonzero weight spaces occur in weights $\{\zeta, \zeta + 2, \zeta+4, \dots\}$. V_ζ is killed by m. Again, (9) shows that the Casimir element $\mathcal{C}$ acts by $\zeta(\zeta-2)/2$.

- Doubly infinite modules, in which the weights are of the form $\zeta + 2\mathbf{Z}$ for $\zeta \in \frac{1}{q}\mathbf{Z}$.

Definition 2.1. *For an infinite dimensional irreducible $(\mathfrak{g}, K)$-module V we define the complementary irreducible representation V^{cl} to be*

$$V^{cl} = \begin{cases} \text{the irreducible } (\mathfrak{g}, K)\text{-module with highest weight } \zeta - 2 & V \text{ has lowest weight } \zeta \\ \text{the irreducible } (\mathfrak{g}, K)\text{-module with lowest weight } \zeta + 2 & V \text{ has highest weight } \zeta \\ V & \text{else} \end{cases}$$

In §4 we use the following key fact about $(\mathfrak{g}, K)$-modules.

Proposition 2.2. *Let V be an irreducible infinite-dimensional $(\mathfrak{g}, K)$-module with infinitesimal character λ. Then, for any $(\mathfrak{g}, K)$-module W:*

(a) *If $\mathcal{C} - \lambda$ is surjective on W, then $\mathrm{Ext}^1_{(\mathfrak{g},K)}(V, W) = 0$.*

(b) *If W is irreducible, $\mathrm{Ext}^1_{(\mathfrak{g},K)}(V, W)$ is one-dimensional if $W \simeq V^{cl}$, and is zero otherwise.*

Proof. We will prove these statements in the case where V is a lowest weight module, which is the case of our main application. The same proof works with slight modifications for V a highest weight or doubly infinite module: in every case, one takes an arbitrary lift of a generating vector, and modifies it using the surjectivity of an appropriate operator.

We prove (a). Take V to be generated by a vector v_ζ of lowest weight ζ with $mv_\zeta = 0$. This implies by the classification above that

$$\lambda = \frac{\zeta(\zeta - 2)}{2}. \tag{11}$$

Take an extension $W \to E \to V$; to give a splitting we must lift v_ζ to a vector in E of K-type ζ killed by m. Arbitrarily lift v_ζ to $\tilde{v}_\zeta \in E_\chi$. Then $m\tilde{v}_\zeta \in W_{\zeta-2}$ and it suffices to show that it lies inside the image of $m : W_\zeta \to W_{\zeta-2}$, for we then modify the choice of $\tilde{v}_\zeta$ by any preimage to get the desired splitting. By (9) and (11) we see that $\mathcal{C} - \lambda : W_{\zeta-2} \to W_{\zeta-2}$ agrees with $2mp$. Since it is surjective, it follows that in particular $m : W_\zeta \to W_{\zeta-2}$ is surjective.

We pass to (b). Suppose W is irreducible; then $\mathrm{Ext}^1_{(\mathfrak{g},K)}(V, W)$ vanishes unless W has the same $\mathcal{C}$-eigenvalue as V. The argument above exhibits an injection of

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(V, W) \hookrightarrow W_{\zeta-2}/mW_\zeta$$

and inspection of K-types amongst those irreducibles with the same $\mathcal{C}$-eigenvalue as W shows that this also vanishes unless $W \simeq V^{cl}$, in which case it is one-dimensional. It remains only to exhibit a nontrivial extension of V by V^{cl}, which is readily done by explicit computation. □

2.4. **Globalizations.** A globalization of a $(\mathfrak{g}, K)$-module V is any continuous G-representation on a topological vector space $\overline{V}$ such that $\left(\overline{V}\right)_K = V$. We will consider two instances of this: the *smooth, or Casselman-Wallach globalization* V_∞, and the *distributional globalization* $V_{-\infty}$.

Following [Cas89], the representation V_∞ is the unique globalization of V as a moderate growth Fréchet G-representation. By definition, such a representation is a Fréchet space F (necessarily topologized with respect to a family of seminorms) such that for any seminnorm $\|\cdot\|_\alpha$, there is an integer N_α and a seminorm $\|\cdot\|_\beta$ for which

$$\|gv\|_\alpha \le \|g\|^{N_\alpha}\|v\|_\beta.$$

The distributional globalization is a dual notion. Indeed, denote by V^* the K-finite part of the dual of V, equipped with the contragredient $(\mathfrak{g},K)$-module structure. Then

$$(V_\infty)^* = (V^*)_{-\infty} \tag{12}$$

where on the left-hand side, the dual is understood as continuous.

We recall an explicit construction of $V_{-\infty}$, see [BO98, §2-3], although it will not be directly used in the rest of the paper: Given V^* as above, and let $W^* \subset V^*$ be a finite-dimensional K-stable subspace that generates V^* as a $(\mathfrak{g},K)$-module. Let $(W^*)^* =: W \subset V$ viewed as a K-representation, and consider the space

$$\mathcal{E}_W = \Big\{ f \in C^\infty(G,W) \mid f(gk) = k^{-1}f(g),\, g\in G,\, k\in K\Big\}.$$

Then the image of V under the map $i: V \to \mathcal{E}_W$ characterized by $\langle i(v)(g), w^*\rangle := \langle v, gw^*\rangle$ $(v\in V,\ w^*\in W^*)$ belongs to the space $\mathcal{A}_W^G$ of sections of moderate growth, i.e. of functions $f\in\mathcal{E}_W$ such that for every differential operator $X\in U(\mathfrak{g})$, there is $R = R(f,X)$ for which

$$\|f\|_{X,R} = \sup_{g\in G}\frac{|Xf(g)|}{\|g\|^R} < \infty. \tag{13}$$

We note that this differs from the notion of *uniform* moderate growth, where one requires R to be taken independently of X.

The space $\mathcal{A}_W^G$ is topologized as the direct limit of Fréchet spaces with respect to the seminorms $\|\cdot\|_{X,R}$. The map i is injective since W^* generates V^*, and the distributional globalization is defined as

$$V_{-\infty} := \overline{i(V)} \subset \mathcal{S}_\infty\mathcal{E}_0.$$

3. First proof of Theorem 1.2: resolutions of principal series.

In this section, we derive Theorem 1.2 from the results of Bunke–Olbrich [BO98], adapting the arguments of Section 9 therein to non-spherical principal series. The two essential ingredients of this argument are the following points established by Bunke–Olbrich, which we shall use as "black boxes":

- acyclicity of Γ acting on spaces of moderate growth functions on G/K, and
- surjectivity of a Laplace-type operator acting on these spaces.

Fix a Casimir eigenvalue λ, and a lattice $\Gamma\subset G$. Given ζ a one-dimensional representation of K, define the following spaces of smooth functions (compare with 2.4, and see (13) in particular for the notion of moderate growth, which is *not* the same as uniform moderate growth):

$$\begin{aligned}
\mathcal{A}^G,\ (\text{resp. } \mathcal{A}) &= \text{moderate growth functions on } G \text{ (resp. on } \Gamma\backslash G).\\
\mathcal{A}^G_\zeta,\ \mathcal{A}_\zeta &= \text{subspace with right } K\text{-type } \zeta\text{:}\quad f(gk) = f(g)\zeta(k).\\
\mathcal{A}^G_\zeta(\lambda),\ \mathcal{A}_\zeta(\lambda) &= \text{subspace with right } K\text{-type } \zeta \text{ and Casimir eigenvalue } \lambda.\\
\mathsf{Cusp}_\zeta(\lambda) &= \text{ subspace of } \mathcal{A}_\zeta(\lambda) \text{ consisting of cuspforms.}
\end{aligned} \tag{14}$$

We will first prove a variant of Theorem 1.2 for principal series. Let B be the preimage of the upper triangular matrices inside G; we may write

$$B = MAN$$

where A and N are as in (10), and $M = Z_K(A) \simeq \mathbf{Z}/2q\mathbf{Z}$. Denote by $\xi \in \mathbf{C}$ the character of A sending $a(y) \mapsto y^{\xi}$. Given a pair of characters (σ, ξ) of M and A respectively let

$$H = \{f : G \to \mathbf{C} \mid f(mang) = a^{\xi-1}\sigma^{-1}(m)f(g), f\ K\text{-finite}\} \tag{15}$$

be the Harish-Chandra module of K-finite vectors in the corresponding principal series representation. This depends on σ and ξ, but to simplify the notation we will not include them explicitly.

Denote by $H_{-\infty}$ its distributional completion (§2.4); explicitly, if we identify H as above with functions on K which transform on the left under the character σ^{-1}, then $H_{-\infty}$ is the corresponding space of *distributions* on K.

In the following proposition, we will assume that H is either irreducible, or decomposes as

$$0 \to \bar{V} \to H \to V \to 0. \tag{16}$$

where both the subrepresentation and quotient are irreducible $(\mathfrak{g}, K)$-modules.

Proposition 3.1. *Let G be the degree q connected cover of $SL_2(\mathbf{R})$. Denote by λ the eigenvalue by which $\mathcal{C}$ acts on H_K; then there are natural isomorphisms*

$$\begin{aligned} H^0(\Gamma, H_{-\infty}) &\simeq \mathcal{A}_\zeta(\lambda) \\ H^1(\Gamma, H_{-\infty}) &\simeq \mathsf{Cusp}_\zeta(\lambda) \\ H^i(\Gamma, H_{-\infty}) &= 0 \qquad \text{for } i \geq 2. \end{aligned}$$

where ζ is any K-weight generating the dual $(\mathfrak{g}, K)$-module H^* (equivalently: ζ belongs to the K-weights of the largest irreducible quotient of that dual, i.e. to $\bar{V}^*$ or to H^* according to whether H^* is reducible or not.)

Proof. In §9 of [BO98] this result is proven in the case of $q = 1$ and the trivial K-type. We will outline the argument to make clear that it remains valid in the situation where we now work, i.e., permitting a covering of $\mathrm{SL}_2(\mathbf{R})$ and an arbitrary K-type.

Fix $v^* \in H^*$ of K-type ζ. Then the rule sending $\mathcal{D} \in H_{-\infty}$ to the function $\mathcal{D}(gv^*)$ on G induces an *isomorphism*

$$H_{-\infty} \simeq \mathcal{A}_\zeta^G(\lambda). \tag{17}$$

We will outline a direct proof of this isomorphism. Injectivity, at least, follows readily: if $\mathcal{D}$ lies in the kernel, it would annihilate the $(\mathfrak{g}, K)$-module grenerated by v^*, which is all of H^*, and by continuity $\mathcal{D}$ is then zero.

Surjectivity is less formal. First one checks finiteness on K-finite functions: one must check that a function f of fixed right- and left- K-type, and with a specified Casimir eigenvalue, occurs in the image of the map above. However, f is uniquely specified up to constants by this property: using the decomposition $G = KAK$, the Casimir eigenvalue amounts to a second order differential equation for the function $y \mapsto f(a_y)$ for $y \in (1, \infty)$, and of the two-dimensional space of solutions only a one-dimensional subspace extends smoothly over $y = 1$; see [Kit17, p. 12-13] for an explicit description both of the differential equation and a hypergeometric basis for the solutions.[1] It follows from this uniqueness that f must agree with $\mathcal{D}(gv^*)$ where $\mathcal{D}$ and v^* match the left- and right- K-types of f. To pass from surjectivity onto K-finite vectors to surjectivity, we take arbitrary $f \in \mathcal{A}_\zeta^G(\lambda)$ and expands as a sum $\sum_\xi f_\xi$ of left K-types. Each f_ξ has a preimage v_ξ according to the previous argument; so one must verify that $\sum_\xi v_\xi$ converges inside $H_{-\infty}$, and for this it is enough to show that $\|v_\xi\|$ grows polynomially with respect to $|\xi|$ (here we compute $\|v_\xi\|$ as the L^2-norm restricted to K in (15)). For this one must "effectivize" the previous argument: The moderate growth property of f implies a bound of the form $|f_\xi(g)| \le c\|g\|^N$, uniform in ξ. On the other hand, $f_\xi = v_\xi(gv^*)$, and such a matrix coefficient always is not too small:

$$|v_\xi(gv^*)| \ge (1+|\xi|)^{-M}\|v_\xi\| \text{ for some } \|g\| \le (1+|\xi|)^M. \tag{18}$$

Such lower estimates on matrix coefficients can be obtained by keeping track of error bounds in asymptotic expressions. They are developed in greater generality in the Casselman–Wallach theory, see e.g. Corollary 12.4 of [BK14] for a closely related result. Combining (18) with the upper bound on f_ξ shows that $\|v_\xi\| \le c(1+|\xi|)^{MN+M}$ as desired.

Therefore, $H_{-\infty}$ is the kernel of

$$\mathcal{A}_\zeta^G \xrightarrow{\mathcal{C}-\lambda} \mathcal{A}_\zeta^G, \tag{19}$$

in the notation of (14). We now invoke surjectivity of a Laplace operator: the morphism $\mathcal{C}-\lambda$ of (20) is surjective, by [BO98, Theorem 2.1]; and consequently (20) is in fact a resolution of $H_{-\infty}$. Moreover, [BO98, Theorem 5.6] asserts that the higher cohomology of Γ acting on $\mathcal{A}_\zeta^G$ vanishes; it is for this argument that Bunke and Olbrich use "moderate growth" rather than "uniform moderate growth." Consequently, the Γ-cohomology of $H_{-\infty}$ can be computed by taking Γ-invariants on the complex (20):

$$(\mathcal{A}_\zeta^G)^\Gamma \xrightarrow{\mathcal{C}-\lambda} (\mathcal{A}_\zeta^G)^\Gamma. \tag{20}$$

Clearly, the H^0 here coincides with $\mathcal{A}_\zeta(\lambda)$. On the other hand, the image of $\mathcal{C}-\lambda$ contains the orthogonal complement of cusp forms (see [BO98, Thm. 6.3], cf. Proposition 4.1), and so the H^1 coincides with the cokernel of $\mathcal{C}-\lambda$ acting on cusp forms; there we can pass to the orthogonal complement and identify $H^1 \simeq \mathsf{Cusp}_\zeta(\lambda)$ as desired.[2] □

The following lemma will be useful in the sequel. We omit the proof.

Lemma 3.2. *Let ζ be, as in Proposition 3.1, a K-weight on H^* which generates the latter as $(\mathfrak{g},K)$-module; fix $v_\zeta \in H^*$ nonzero of weight ζ. For any $(\mathfrak{g},K)$-module V, there is an isomorphism*

$$\mathrm{Hom}_{(\mathfrak{g},K)}(H^*, V) \to V_\zeta(\lambda), \quad f \mapsto f(v_\zeta), \tag{21}$$

where $V_\zeta(\lambda)$ is the subspace of V_ζ killed by $\mathcal{C}-\lambda$.

For reducible principal series as in (16), we prove:

[1] There are other references in the mathematical literature but Kitaev explicitly considers the universal cover.

[2] In fact, $\mathcal{C}-\lambda$ is adjoint to $\mathcal{C}-\bar{\lambda}$, but the kernel of the latter of either is only nonzero if λ is real, so we do not keep track of the complex conjugate.

Proposition 3.3. *Let $H_{-\infty}$ with infinitesimal character λ decompose as in* (16)*. Then*

$$H^1(\Gamma, V_{-\infty}) \simeq H^1(\Gamma, H_{-\infty}) \simeq \mathsf{Cusp}_\zeta(\lambda).$$

Proof. We will deduce the result from Proposition 3.1 together with the long exact sequence in cohomology associated to (the distributional globalization of) (16) and that of the principal series $\bar{H}$ such that

$$0 \to V \to \bar{H} \to \bar{V} \to 0, \tag{22}$$

i.e. for which the roles of subrepresentation and quotient are swapped between $\bar{V}$ and V. Such a principal series can be obtained by replacing the role of upper triangular matrices by lower triangular matrices above. We first consider the long exact sequence associated to (the distribution globalization of) (16), namely:

$$\begin{aligned} 0 \to H^0(\Gamma, \bar{V}_{-\infty}) \to H^0(\Gamma, H_{-\infty}) \xrightarrow{\Omega} H^0(\Gamma, V_{-\infty}) \to \\ \to H^1(\Gamma, \bar{V}_{-\infty}) \to H^1(\Gamma, H_{-\infty}) \xrightarrow{\Pi} H^1(\Gamma, V_{-\infty}) \to 0. \end{aligned} \tag{23}$$

We have used here that the next group $H^2(\Gamma, \bar{V}_{-\infty})$ of the sequence vanishes: it is isomorphic to $H^3(\Gamma, V_{-\infty})$ by the long exact sequence associated to (22) and Proposition 3.1, and that H^3 vanishes always. Indeed, let $\overline{\Gamma}$ be the image of $\Gamma \to \mathrm{SL}_2(\mathbf{R})$, and $\mu \leqslant \Gamma$ the kernel of $\Gamma \to \overline{\Gamma}$; if V is a $\mathbf{C}[\Gamma]$-module then $H^i(\Gamma, V) = H^i(\overline{\Gamma}, V^\mu)$, and, being a lattice in $\mathrm{SL}_2(\mathbf{R})$, the virtual cohomological dimension of $\overline{\Gamma}$ is at most 2.

We claim that the penultimate map Π of (23) is an isomorphism; this will conclude the proof, using Proposition 3.1. Our claim will follow if we can prove that

$$\dim \text{cokernel}\, \Omega \geq \dim H^1(\Gamma, \bar{V}_{-\infty}).$$

By applying Proposition 3.1 to $\bar{H}$, we find that $H^1(\Gamma, \bar{V}_{-\infty})$ is a quotient of $\mathsf{Cusp}_\chi(\lambda)$, for χ a weight in V^*. It therefore suffices to show that

$$\dim\ \text{cokernel}\, \Omega \geq \dim \mathsf{Cusp}_\chi(\lambda). \tag{24}$$

The codomain $H^0(\Gamma, V_{-\infty})$ of Ω is identified, by means of Frobenius reciprocity (2), with the space of homomorphisms from the dual $(\mathfrak{g}, K)$ module V^* to the K-finite vectors $\mathcal{A}_K$ in the space of automorphic forms. Contained in this space is the corresponding space $H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}}$ of *cuspidal* homomorphisms, that is to say, those homomorphisms that are valued in cusp forms. Note that all homomorphisms from $\bar{H}^*$ to the space of cusp forms factor through V^* by semisimplicity of the space of cusp forms (which in turn follows by unitarity). Applying Lemma 3.2 to $\bar{H}$, then, identifies $H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}}$ with $\mathsf{Cusp}_\chi(\lambda)$. Therefore, (24) is equivalent to the assertion that dim cokernel $\Omega \geq \dim H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}}$; so it is enough to check that $H^0(\Gamma, V_{-\infty})^{\mathrm{cusp}}$ intersects trivially the image of Ω.

We must prove, then, that no homomorphism from V^* to Cusp_K can be extended to a homomorphism from H^* to $\mathcal{A}_K$. Suppose, then, that $f : H^* \to \mathcal{A}_K$ is a $(\mathfrak{g}, K)$-module homomorphism whose restriction to V^* is nonzero and has cuspidal image. We now make use of the orthogonal projection map from all automorphic forms to cusp forms, which exists because one can sensibly take the inner product of a cusp form with any function of moderate growth. Post-composing f with this projection gives a morphism from H^* to the semisimple $(\mathfrak{g}, K)$-module Cusp_K; since

H^* is a nontrivial extension of $\bar{V}^*$ by V^*, this morphism is necessarily trivial on the subrepresentation V^*, a contradiction. □

Now let us deduce Theorem 1.2. We divide into cases according to how the representation W of the theorem can be fit into a principal series:

- W is an irreducible principal series, equivalently, W is doubly-infinite. In this case, $W^{cl} = W$, and combining Proposition 3.1 and Lemma 3.2 gives the statement of Theorem 1.2.
- W is an irreducible subquotient of a principal series with two composition factors. In this case we can suppose that $W = V^*$ with notation as in (16). In that notation we have $W^* = V$, and $W^{cl} = \bar{V}^*$. Proposition 3.3 gives $H^1(\Gamma, V_{-\infty}) \simeq \mathsf{Cusp}_\zeta(\lambda)$, and Lemma 3.2 shows that $\mathsf{Cusp}_\zeta(\lambda)$ is identified with the space of $(\mathfrak{g}, K)$-homomorphisms from H^* to the space of cusp forms; by semisimplicity of the target such a homomorphism factors through the irreducible quotient $\bar{V}^* = W^{cl}$. This proves Theorem 1.2 in this case.
- W is an irreducible subquotient of a principal series with more than two composition factors. In this case, W is necessarily a highest- or lowest-weight module factoring through $\mathrm{SL}_2(\mathbf{R})$, and there is an exact sequence

$$F \to H \to \mathcal{D}, \tag{25}$$

  where F is finite-dimensional and $\mathcal{D}$ is the sum of W^* and another highest- or lowest- weight module. Here, $W^{cl} = F$ and Theorem 1.2 is equivalent to the vanishing of $H^1(\Gamma, W^*_{-\infty})$. In the case of a discrete series that factors through $\mathrm{PSL}_2(\mathbf{R})$, this vanishing follows from [BO98, Prop. 8.2], and the remaining case of an "odd" discrete series is handled by the same argument. Namely, use the long exact sequence associated to (25); the argument of Proposition 3.1 shows that $H^1(\Gamma, H_{-\infty}) = 0$, and also $H^2(\Gamma, F) = 0$, so also $H^1(\Gamma, \mathcal{D}_{-\infty}) = 0$ and so its summand $H^1(\Gamma, W^*_{-\infty})$ also vanishes.

4. Second proof of Theorem 1.2: extensions of $(\mathfrak{g}, K)$-modules

Our original proof of Theorem 1.2 proceeds by a reduction to a computation in the category of $(\mathfrak{g}, K)$ modules. The two essential ingredients of this argument are the following points:

(a) the Casselman–Wallach theory which gives a canonical equivalence between suitable categories of *topological* G-representations and *algebraic* $(\mathfrak{g}, K)$-modules.

(b) surjectivity of a Laplace-type operator acting, now, on spaces of moderate growth functions on $\Gamma\backslash G$;

We will not prove (a), although we will briefly sketch an elementary proof of what we use from it. We will prove (b) in the next section.

Let λ be the eigenvalue by which the Casimir $\mathcal{C} \in Z(\mathfrak{g})$ of (9) acts on W (the irreducible $(\mathfrak{g}, K)$-module from the statement of Theorem 1.2). We will use the notation $\mathcal{A}$ from (14) for the space of smooth, *uniform* moderate growth functions on $\Gamma\backslash G$, which is to say that there exists R such that for all $X \in \mathfrak{U}$,

$$\|f\|_{X,R} = \sup_{g \in G} \frac{|Xf(g)|}{\|g\|^R} < \infty. \tag{26}$$

(compare with (13), and beware that we are using the same notation as in Section 3, but for a slightly different space). We use *uniform* moderate growth because it interfaces more readily with the Casselman–Wallach theory; by contrast, §3 used moderate growth because this is used in the acyclicity result mentioned after (20).

Also consider following the subspaces of $\mathcal{A}$:

$$\begin{aligned}\mathcal{A}_{\lambda-\mathrm{nil}} &= K\text{-finite functions on which } \mathcal{C}-\lambda \text{ acts nilpotently},\\ \mathsf{Cusp}(\lambda) &= \text{subspace of } \mathcal{A}_{\lambda-\mathrm{nil}} \text{ consisting of cusp forms.}\end{aligned}$$

The precise form of (b) we will use is this:

Proposition 4.1. *The image of the map $\mathcal{C}-\lambda : \mathcal{A}_K \to \mathcal{A}_K$ is precisely the subspace of $\mathcal{A}_K$ orthogonal to* $\mathsf{Cusp}(\lambda)$.

This is almost [BO98, Theorem 6.3], except there the argument is for moderate growth functions rather than uniform moderate growth; they state on *op. cit.* p. 73 that the same proof remains valid in the uniform moderate growth. Also, Cassleman proves [Cas84, Theorem 4.4], for the trivial K-type, that $\mathcal{C}$ is surjective on spaces of Eisenstein distributions, from which a similar result can be extracted. Since the above statement is in a sense the crux of the argument, and neither reference gives it in precisely this form, we have given a self-contained proof in §5. Our proof follows a slightly different strategy and is perhaps of independent interest.

4.1. **Proof of Theorem 1.2: reduction to $(\mathfrak{g}, K)$ extensions.** We begin the proof of Theorem 1.2 assuming Proposition 4.1. We begin by constructing an isomorphism

$$H^1(\Gamma, W_{-\infty}^*) \simeq \mathrm{Ext}^1_G(W_\infty, \mathcal{A}), \tag{27}$$

where W_∞ is the smooth globalization of W.

On the left, we have the ordinary group cohomology of the discrete group Γ acting on the vector space $W_{-\infty}^*$, without reference to topology. On the right here we use a *topological* version of Ext defined as follows: present $\mathcal{A}$ as a directed union $\varinjlim \mathcal{A}(R)$ of moderate growth Fréchet G-representations (see§2.4) obtained by imposing a specific exponent of growth R in (26). The right hand side is then defined to be the direct limit $\varinjlim \mathrm{Ext}^1_G(W_\infty, \mathcal{A}(R))$, where the elements of each Ext group are represented by isomorphism classes of short exact sequences[3] $\mathcal{A}(R) \to ? \to W_\infty$, with ? a moderate growth Fréchet G-representation and the maps are required to be continuous.

The statement (27) is then a version of Shapiro's lemma in group cohomology. Let us spell out the relationship: for $G_1 \leq G_2$ of finite index, and W a finite-dimensional G_1-representation, Shapiro's lemma supplies an isomorphism

$$H^1(G_1, W^*) \stackrel{(a)}{\simeq} H^1(G_2, \mathrm{I}_{G_1}^{G_2} W^*) \stackrel{(b)}{\simeq} H^1(G_2, (\mathrm{I}_{G_1}^{G_2}\mathbf{C}) \otimes W^*) \stackrel{(c)}{\simeq} \mathrm{Ext}^1_{G_2}(W, \mathrm{I}_{G_1}^{G_2}\mathbf{C}). \tag{28}$$

Here $I_{G_1}^{G_2}$ is the induction from G_1 to G_2, and we used in (a) Shapiro's lemma in its standard form; at step (b) the projection formula $\mathrm{I}_{G_1}^{G_2} W^* \simeq \mathrm{I}_{G_1}^{G_2}\mathbf{C} \otimes W^*$, and at step (c) the relationship between group cohomology and Ext-groups which results by deriving the relationship $\mathrm{Hom}_{G_2}(W, V) = (V \otimes W^*)^{G_2}$.

[3]Here, the notion of "exact sequence" is the usual one - there is no reference to the topology.

Our statement (27) is precisely analogous to the isomorphism of (28) with Γ playing the role of G_1, G playing the role of G_2, and with topology inserted. It can be proven simply by writing down the explicit maps from far left to far right in (28) and checking that they respect topology and are inverse to one another. There is only one point that is not formal: to prove that there is a well-defined map from left to right, one needs to check that the extension of G-representations arising by "inflating" a cocycle $j : \Gamma \to W^*_{-\infty}$ indeed has moderate growth. This requires growth bounds on j, and these follow by writing out $j(\gamma)$, for arbitrary $\gamma \in \Gamma$, in terms of the values of j on a generating set using the cocycle relation.We observe that some "automatic continuity" argument of this nature is needed, because, in the statement of (27), the topology of W figures on the right hand side but not on the left.

Next, observe that there is a natural map

$$\mathrm{Ext}^1_G(W_\infty, \mathcal{A}) \longrightarrow \mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}}), \tag{29}$$

where the right-hand side is taken in the category of $(\mathfrak{g}, K)$-modules. This "natural map" associates to an extension $\mathcal{A} \to E \to W_\infty$ the underlying sequence of K-finite vectors in each of $\mathcal{A}, E, W_\infty$ which are annihilated by some power of $\mathcal{C} - \lambda$; that the resulting sequence remains exact follows from surjectivity of $\mathcal{C} - \lambda$ in the form of Proposition 4.1. One must verify that each element $w \in W$ killed by $(\mathcal{C} - \lambda)^n$ has a preimage in E_K of the same type. To do this, choose an arbitrary preimage e; then $(\mathcal{C} - \lambda)^n e$ belongs to the image of $\mathcal{A}_K$, and can be written as $f_1 + f_2$ with $f_1 \in \mathsf{Cusp}(\lambda)$ and $f_2 \in \mathsf{Cusp}(\lambda)^\perp$. Choose, by Proposition 4.1, a class $e' \in \mathcal{A}_K$ with $(\mathcal{C} - \lambda)^n e' = f_2$; then $e - e'$ still lifts w and is now killed by $(\mathcal{C} - \lambda)^{n+1}$.

Next, we show that the map of (29) is in fact an *isomorphism*. Injectivity of the resulting map on Ext-groups follows using the Casselman-Wallach theory of canonical globalization; the result is formulated in exactly the form we need in [BK14, Prop 11.2], namely, a splitting at the level of $(\mathfrak{g}, K)$-modules automatically gives rise to a continuous splitting. [4] Surjectivity does not follow at once from the Casselman–Wallach theory because $\mathcal{A}$ is "too big," however, in the case at hand, it can be checked directly, because we will see below that all the $(\mathfrak{g}, K)$-exts are induced from a certain finite length direct summand Cusp_λ to which the results of [Cas89] (in the form of the equivalence of categories, see [Wal92, Corollary, §11.6.8]) can be applied.

Remark 4.2. *Together, the isomorphisms* (27) *and* (29) *give an isomorphism*

$$H^1(\Gamma, W^*_{-\infty}) \simeq \mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-nil}). \tag{30}$$

The analogous statement for Γ *cocompact has been proved in much greater generality – for all* Exts, *by Bunke-Olbrich* [BO97, Theorem 1.4]. *However, our argument does not generalize, at least in any routine way, to higher* Exts*: it is not immediately clear to us how to generalize the cocycle growth argument to* H^i *for* $i > 1$.

---

[4] We sketch the idea of the argument to emphasize that what we use is relatively elementary: Given an abstract $(\mathfrak{g}, K)$-module splitting $W \to \mathcal{A}$ we must show that it does not distort norms too far. Fixing a generating set $w_1, \ldots, w_r$ for W, one shows using bounds similar to (18) that any vector $w \in W$ can be written as $\sum h_i \star w_i$ where h_i are bi-K-finite functions on G and the norms of the h_i are not too large in terms of the norms of w. This permits one to bound the size of $\varphi(w) = \sum h_i \star \varphi(w_i)$.

4.2. **Evaluation of the $(\mathfrak{g}, K)$-ext.** No conclude the proof of Theorem 1.2, we now compute the $(\mathfrak{g}, K)$-extension on the right-hand side of (30). The space $\mathsf{Cusp}(\lambda)$ decomposes as a finite direct sum of irreducible $(\mathfrak{g}, K)$-modules; this follows from the similar L^2 statement, see [Bor97, §8]. Since each of these irreducible summands has infinitesimal character λ, their underlying $(\mathfrak{g}, K)$-modules can belong to at most three isomorphism classes, as described in §2.3; among these is W^{cl}, the "complementary $(\mathfrak{g}, K)$-module to W" from Definition 2.1.

Accordingly we decompose

$$\mathcal{A}_{\lambda-\mathrm{nil}} = \mathsf{Cusp}(\lambda)^{\perp} \oplus (W^{cl})^m \oplus \bigoplus_{\substack{V \subset \mathsf{Cusp}(\lambda) \\ V \not\simeq W^{cl}}} V, \tag{31}$$

where $\mathsf{Cusp}(\lambda)^{\perp}$ is the orthogonal complement of $\mathsf{Cusp}(\lambda)$ within $\mathcal{A}_{\lambda-\mathrm{nil}}$, and m is the multiplicity of W^{cl} in $\mathsf{Cusp}(\lambda)$.

The splitting (31) induces a similar direct sum splitting of $\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}})$. But Proposition 4.1 implies that $\mathcal{C} - \lambda$ defines a surjection from $\mathsf{Cusp}(\lambda)^{\perp}$ to itself, and so, applying Proposition 2.2,

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathsf{Cusp}(\lambda)^{\perp}) = 0.$$

The remaining two summands evaluate via the second part of Proposition 2.2 to $\mathbf{C}^m$ and 0 respectively. This yields

$$\mathrm{Ext}^1_{(\mathfrak{g},K)}(W, \mathcal{A}_{\lambda-\mathrm{nil}}) \simeq \mathbf{C}^m,$$

which concludes the proof, remembering that (27) and (29) identified the left-hand side with $H^1(\Gamma, W^*_{-\infty})$ and that m is the multiplicity of W^{cl} in the space of cusp forms.

5. Surjectivity of Casimir on the space of automorphic forms.

The primary analytic ingredient in both proofs is the surjectivity of a Laplacian-type operator; in the first proof this is used both on G and on $\Gamma\backslash G$, and in the second proof it is used only on $\Gamma\backslash G$. We will now give a self-contained proof of the second version, Proposition 4.1. As noted after that Proposition, this statement is essentially in the literature, but given its importance it seemed appropriate to give a self-contained proof.

We follow here the notation of §4; in particular, $\mathcal{A}$ is defined using the notion of *uniform* moderate growth. It is enough to show that every function orthogonal to $\mathsf{Cusp}(\lambda)$ occurs in the image of $\mathcal{C} - \lambda : \mathcal{A}_K \to \mathcal{A}_K$. The basic strategy is as follows:

- (i) In §5.4, we decompose elements of $\mathcal{A}_K$ into functions "near the cusp" and functions of rapid decay, and
- (ii) in §5.5, we construct preimages under $\mathcal{C} - \lambda$ for functions in each subspace. Doing this "near the cusp" amounts to solving an ODE; the construction of preimages for functions of rapid decay is carried out via L^2-spectral theory.

Since $\mathcal{C} - \lambda$ commutes with K, it suffices to prove the Proposition with $\mathcal{A}_K$ replaced by its subspace $\mathcal{A}_\zeta$ with K-type ζ. In what follows, we will regard ζ as fixed.

5.1. **Cusps.** It is convenient to fix once and for all a fundamental domain for $\Gamma\backslash G$: we take

$$\digamma = \{z \in \mathbb{H} : d(z,i) \leq d(\gamma z, i) \text{ for all } \gamma \in \Gamma - \{e\}\}, \tag{32}$$

which describes a convex hyperbolic polygon which is (up to boundary) a fundamental domain for Γ acting on $\mathbb{H}$; its pullback to G via $g \mapsto g \cdot i$ is a fundamental domain for $\Gamma\backslash G$, which will often be denoted by the same letter. In particular, $\digamma$ can be decomposed in the following way, where the sets intersect only along their boundary:

$$\digamma = \digamma_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2 \cup \cdots \cup \mathcal{C}_h \tag{33}$$

with $\digamma_0$ compact and each $\mathcal{C}_i$ a *cusp*, that is to say, a G-translate of a region of the form $\{x+iy : a \leq x \leq b, y \geq Y_0\}$. In the Iwasawa cordinates $G = NAK$, the pullback of $\mathcal{C}_i$ to G therefore has the form

$$\widetilde{\mathcal{C}_i} = g_i \cdot \{n_x a_y k : a \leq x \leq b, y \geq Y_0, k \in K\}. \tag{34}$$

The map $\widetilde{\mathcal{C}_i} \to \Gamma\backslash G$ is injective on the interior of $\widetilde{\mathcal{C}_i}$. We will often identify $\widetilde{\mathcal{C}_i}$ with its image in $\Gamma\backslash G$.

5.2. **The constant term and moderate growth functions in the cusp.** Let $f \in \mathcal{A}_K$. Fix a cusp i; we write Γ_i^N for $\Gamma \cap g_i N g_i^{-1}$. The constant term $f_i^N : g_i N g_i^{-1}\backslash G \longrightarrow \mathbf{C}$ is defined by the rule

$$f_i^N : x \mapsto \text{average value of } f(g_i n_t g_i^{-1} x) \text{ for } t \in \mathbf{R}. \tag{35}$$

The function $f(g_i n_t g_i^{-1} x)$ is periodic in t and therefore the notion of its average value makes sense. Moreover, the above map is right G-equivariant. A basic (and elementary) fact is that f_i^N is asymptotic to f inside $\widetilde{\mathcal{C}_i}$; indeed the function $f - f_i^N$ has rapid decay in $\widetilde{\mathcal{C}_i}$, as proved in [Bor97, 7.5]. Here, we say that a function $J : \widetilde{\mathcal{C}_i} \to \mathbf{C}$ has *rapid decay* if, for any $X_1, \ldots, X_r \in \mathfrak{g}$ and any positive integer N we have

$$\sup_{\widetilde{\mathcal{C}_i}} \|g\|^N \, |X_1 \ldots X_r J(g)| < \infty. \tag{36}$$

Let us consider more generally functions on G that are left N-invariant and have fixed right K-type ξ.

Such a function may be identified, by means of pullback by $y \mapsto a_y$, with functions on $\mathbf{R}_+$. The condition that such a function has finite norm under $\|\cdot\|_{X,R}$ for all X, with notation as in (26), is equivalent to asking that

$$\left|(y\frac{d}{dy})^j f\right| < C_j \cdot (|y|^{-1} + |y|)^R \tag{37}$$

for all j. That this condition is necessary is seen by applying (26) to $X_i \in \mathrm{Lie}(A)$. To see that it is sufficient, we fix $\mathfrak{U}$ belonging to the universal enveloping algebra; now, for any $k \in K$, we may write $\mathfrak{U}$ as a sum of terms $\sum c_i(k)(\mathrm{Ad}(k^{-1})\mathfrak{U}_{N,i})(\mathrm{Ad}(k^{-1})\mathfrak{U}_{A,i})\mathfrak{U}_{K,i}$ where the terms belong to fixed bases for the universal enveloping algebra of N, A and K respectively, and the coefficients $c_i(k)$ are bounded independently of k. This permits us to bound $\mathfrak{U}f(nak)$ and we see that the bound (37) suffices.

This motivates the following definition: Fix $Y_0 > 0$ and denote by $\mathcal{P}_{\geq Y_0}$ the space of smooth functions on $\mathbf{R}$ supported in $y > Y_0$ satisfying (37) for some R. Because of the restriction that $y > Y_0$, this is equivalent to ask that all its derivatives are "uniformly" polynomially bounded, i.e. there is R such that, for all j, there exists a constant C_j with

$$\left|\frac{d^j f}{dy^j}\right| < C_j(2+|y|)^{R-j}. \tag{38}$$

5.3. **The subspace Eis_λ of Eisenstein series with eigenvalue λ.** To each cusp $\mathcal{C}_j$, we attach an Eisenstein series $E^j(s)$, which is an $\mathcal{A}_\zeta$-valued meromorphic function of the complex variable s, characterized by the fact that for $\mathrm{Re}(s) \gg 1$ it equals:

$$E^j(s,g) = \sum_{\gamma\in\Gamma_N^i\backslash\Gamma} H(g_i^{-1}\gamma g)^s,$$

where H is the unique function on G with right K-type ζ, invariant on the left by N, and on A given by $a_y \mapsto y$.

The resulting vector-valued function is holomorphic when $\mathrm{Re}(s) = 1/2$ and we denote its value at $s = 1/2 + it$ by E_t^j. In words, E_t^j is the unitary Eisenstein series of K-type ζ with parameter $t \in \mathbf{R}$ attached to the jth cusp of $\Gamma\backslash G$. Finally, denoting by λ_t the eigenvalue of $\mathcal{C}$ on E_t^j, let

$$\mathrm{Eis}(\lambda) := \bigoplus_j \left\{ \text{span of all Eisenstein series } E_t^j, \text{ with } t\in\mathbf{R}, \text{ such that } \lambda_t = \lambda \right\},$$

so that $\mathrm{Eis}(\lambda)$ is a finite-dimensional subspace of $\mathcal{A}_\zeta$ annihilated by $\mathcal{C}-\lambda$. However, if $t \mapsto \lambda_t - \lambda$ happens to have a double zero, we include in the above space the derivative $\frac{d}{dt}E_t^j$, for this is also annihilated by $\mathcal{C}-\lambda$. The Casimir eigenvalue of $E^j(s,g)$ is quadratic in s and therefore the dimension of $\mathrm{Eis}(\lambda)$ is at most twice the number of cusps.

5.4. **Decomposition of $\mathcal{A}_\zeta$.** Consider the space of L^2-eigenfunctions of the Laplacian with eigenvalue λ; call this Discrete(λ).

Lemma 5.1. *Let $\widetilde{\mathcal{C}_i}$ be the cusps for a fundamental domain for the action of Γ on G as in* (33)*. Then every $f \in \mathcal{A}_\zeta$, perpendicular to* $\mathsf{Cusp}(\lambda)$*, can be written as the sum*

$$f = f_s + \sum_i f_{c_i}$$

where:

(i) *The function f_s is smooth, has rapid decay at all the cusps, and is perpendicular to* $\mathrm{Eis}(\lambda)\oplus\mathrm{Discrete}(\lambda)$.
(ii) *Each f_{c_i} is supported in the cusp $\widetilde{\mathcal{C}_i}$ and, with reference to the identification* (34)*:*

$$\widetilde{\mathcal{C}_i} = g_i \cdot \{n_x a_y k : a \le x \le b, y \ge Y_0, k \in K\}.$$

has the form $nak \mapsto P(y)\zeta(k)$, where P belongs to the space $\mathcal{P}_{\ge Y_0}$ described after (38).

Although f is only assumed orthogonal to cusp forms, we arrange, in a manner very convenient for our application, that f_s is orthogonal also to $\mathrm{Eis}(\lambda)$ and all of Discrete(λ). This is possible because there is a lot of freedom in the decomposition.

Proof. This is a straightforward cut-off process; the only delicacy is to ensure that f_s is in fact perpendicular to $\mathrm{Eis}(\lambda)$ and $\mathrm{Discrete}(\lambda)$. We start from f_i^N, the constant term along the ith cusp as defined in (35). Take φ_i, ψ_i smooth functions on $\mathbf{R}_+$ where:

- $\varphi_i = 0$ for $y < Y_0$ and $\varphi_i = 1$ for $y > 2Y_0$.
- ψ_i is supported in $(Y_0, 2Y_0)$.

We consider both φ_i and ψ_i as functions on $\widetilde{\mathcal{C}_i}$ described by the rules $g_i n_x a_y k \mapsto \varphi_i(y)\zeta(k)$ and $g_i n_x a_y k \mapsto \psi_i(y)\zeta(k)$ respectively. Now put $f_s = f - \sum_i(\varphi_i f_i^N + \psi_i)$ so that

$$f = f_s + \sum \underbrace{\left(\varphi_i f_i^N + \psi_i\right)}_{f_{c_i}}. \tag{39}$$

We will show that, for suitable choice of ψ_i, (39) is the desired expression. All the properties except perpendicularity to $\mathrm{Discrete}(\lambda) \oplus \mathrm{Eis}(\lambda)$ follow from general properties of the constant term discussed in §5.2. In particular, the uniform bound on P follows from that of f.

Observe that $\varphi_i f_i^N$ and ψ_i are both perpendicular to all cuspidal functions and in particular to $\mathsf{Cusp}(\lambda)$, because they both arise from functions on $g_i N g_i^{-1} \cap \Gamma \backslash G$ which are left invariant by $g_i N g_i^{-1}$. Therefore f_s is also perpendicular to $\mathsf{Cusp}(\lambda)$.

It remains to choose ψ_i in such a way that f_s is indeed perpendicular to the orthogonal complement of $\mathsf{Cusp}(\lambda)$ inside $\mathrm{Discrete}(\lambda) \oplus \mathrm{Eis}(\lambda)$; call this space $\widetilde{\mathrm{Eis}}(\lambda)$, as it is (potentially) a finite-dimensional enlargement of $\mathrm{Eis}(\lambda)$.

To do this, for each $\mathcal{E} \in \widetilde{\mathrm{Eis}}(\lambda)$ we should have

$$\left\langle \sum_i f - \varphi_i f_i^N, \mathcal{E} \right\rangle = \sum_i \left\langle \psi_i, \mathcal{E}_i^N \right\rangle_{\widetilde{\mathcal{C}_i}}.$$

The right-hand side can be considered as a linear mapping from the vector space of possible ψ_i to the finite-dimensional dual $\widetilde{\mathrm{Eis}}(\lambda)^*$ of the vector space $\widetilde{\mathrm{Eis}}(\lambda)$. It is enough to show this mapping is surjective, and for this it is enough to show that its dual is injective. But the dual map is identified with the constant term:

$$\widetilde{\mathrm{Eis}}(\lambda) \to \bigoplus_i C^\infty(T_i, 2T_i), \quad \mathcal{E} \mapsto (\mathcal{E}_i^N)(g_i a_y)$$

and this is injective: if $\mathcal{E}^{N_i}$ vanished in $(T_i, 2T_i)$ then it – being real-analytic – vanishes identically; if this is so for all i, then $\mathcal{E}$ would be a cusp form, contradicting the definition of $\widetilde{\mathrm{Eis}}(\lambda)$. □

5.5. **Surjectivity of $\mathcal{C} - \lambda$.** We now show surjectivity of $\mathcal{C} - \lambda$ on each of the two pieces of $\mathcal{A}_\zeta$ corresponding to the decomposition of Lemma 5.1.

5.5.1. *Surjectivity on the cusp.*

Lemma 5.2. *The operator $\mathcal{C} - \lambda$ is surjective on the space of functions on G which:*

- *are left N-invariant and have fixed right K-type ζ, and*
- *lie in the space $\mathcal{P}_{\geq Y_0}$ described before* (38) *when pulled back to $\mathbf{R}_+$ by means of $y \mapsto a_y$.*

Proof. Let $f : \mathbf{R}_+ \to \mathbf{C}$ be extended to a function $F : G \to \mathbf{C}$ by left N-invariance and with fixed κ-weight equal to ζ, so that F has the form:

$$F(na_y \exp(\theta k)) = f(y)e^{i\zeta\theta}.$$

Observe that for arbitrary $X_1 \in \mathfrak{n} = \mathrm{Lie}(N)$ and $X_2, \dots, X_k \in \mathfrak{g}$ we have

$$(X_1 \dots X_k F) \text{ is identically zero on } NA.$$

Indeed, the left-hand side is the partial derivative $\partial_{t_1} \dots \partial_{t_k}$ of $F(nae^{t_1X_1} \dots e^{t_kX_k})$ evaluated at $t_i = 0$, which vanishes since F is independent of t_1. From this observatoin, it follows that the action of the operator $\mathcal{C} = \frac{H^2}{2} - H + 2XY$ on f agrees with the action of $H^2/2 - H$ on $f(y)$. Since H acts on f via via $2y\frac{d}{dy}$, we get that $\mathcal{C} - \lambda$ acts as the differential operator:

$$2y^2 \frac{d^2}{dy^2} - \lambda.$$

We show that $\mathcal{C}-\lambda$ is surjective on $\mathcal{P}_{\geq Y_0}$ explicitly: we construct g with $(\mathcal{C}-\lambda)g = f$ via the method of variation of parameters.

The homogeneous solutions to the equation $\left(2y^2\frac{d^2}{dy^2} - \lambda\right)g = 0$ are given by y^{p_1}, y^{p_2}, where the p_i are roots of $2p(1-p) + \lambda = 0$. We assume that $p_1 \neq p_2$, the $p_1 = p_2$ case is similar. A solution to $(\mathcal{C} - \lambda)g = f$ can then be found by taking

$$g = a_1(y)y^{p_1} + a_2(y)y^{p_2},$$

where the a_i satisfy

$$\frac{da_i}{dy} = (-1)^i \frac{1/2}{p_1 - p_2} f(y)y^{-p_i-1}.$$

Taking f as in (38), we take $a_i = \pm\frac{p_i-p_2}{2}\int_{Y_0}^{y} f(y)y^{-p_i-1}$ for $y > Y_0$ and $a_i(y) = 0$ for $y \leq Y_0$. By construction, if f belongs to $\mathcal{P}_{\geq Y_0}$ then so does a_i and so also g. □

5.5.2. *Surjectivity on functions of rapid decay.*

Proposition 5.3. *The image of the map $\mathcal{C} - \lambda : \mathcal{A}_\zeta \to \mathcal{A}_\zeta$ contains all functions of rapid decay that are orthogonal to* $\mathrm{Eis}(\lambda)$ *and* $\mathrm{Discrete}(\lambda)$.

Proof. Let f be such a function. We fix an orthonormal basis $\{\varphi_i\}$ for the discrete spectrum of $\mathcal{C} - \lambda$ on $L^2(\Gamma\backslash G)_\zeta$, where the subscript means that we restrict to K-type ζ. For constants μ_j depending only on the width of the various cusps, we have, following e.g. [Bor97, §13],

$$f = \sum_i \langle f, \varphi_i\rangle\varphi_i + \mu_j \sum_j \int_{t\geq 0} \langle f, E_t^j\rangle E_t^j dt. \tag{40}$$

A priori this is an equality inside L^2. Let λ_i and λ_t be, respectively, the eigenvalues of $\mathcal{C} - \lambda$ on φ_i and E_t. Define $\bar{f} \in L^2$ by the rule

$$\bar{f} = \sum_{\lambda_i \neq 0} \frac{\langle f, \varphi_i\rangle}{\lambda_i}\varphi_i + \sum_j \int_{t\in\mathbf{R}} \frac{\langle f, E_t^j\rangle}{\lambda_t} E_t^j dt, \tag{41}$$

(one readily sees that the right hand side is convergent in L^2). We claim that $\bar{f}$ has uniform moderate growth and

$$(\mathcal{C} - \lambda)\bar{f} = f$$

as desired.

In fact, the summation and integrals in both (40) and (41) are absolutely convergent, uniformly on compact sets, and they define functions of uniform moderate growth; moreover, any derivative $X\bar{f}$ coincides with the corresponding summation inserting X inside the sums and integrals. The proof of these claims follow from nontrivial, but relatively standard, estimates. We summarize these estimates, with references. A convenient general reference for all the analysis required is Iwaniec [Iwa95]; it works only with the trivial K-type, but analytical issues are exactly the same if we work with a general K-type.

Let λ_i be the $(\mathcal{C}-\lambda)$-eigenvalue of φ_i. Then the easy upper bound in Weyl's law (cf. [Iwa95, (7.11), Corollary 11.2] for the sharp Weyl law in the spherical case; the same proof applies with K-type) gives:

$$\#\{i : |\lambda_i| \le T\} \le \mathrm{const}\cdot T^2. \tag{42}$$

For any $r \ge 0$ we have an estimate

$$|\langle f, \varphi_i\rangle| \le c_r(1+|\lambda_i|)^{-r} \tag{43}$$

arising from integration by parts and Cauchy-Schwarz (using $\|\varphi_i\|_{L^2} = 1$). Finally, there is a constant N with the following property: for any invariant differential operator $X \in \mathfrak{U}$ of degree d, we have a bound

$$|X\varphi_i(g)| \le (1+|\lambda_i|)^{d+N}\|g\|^N. \tag{44}$$

This can be derived from a Sobolev estimate, again using the normalization $\|\varphi_i\|_{L^2} = 1$; see e.g [BR02, (3.7)]. These estimates suffice to treat the cuspidal summand of (41).

Now we discuss the integral summand of (41). The function $\langle f, E_t\rangle$ is holomorphic in a neighbourhood of $t \in i\mathbf{R}$, as follows from holomorphicity of $t \mapsto E_t$ and absolute convergence of the integral. Moreover, by assumption, this holomorphic function vanishes when $\lambda_t = 0$. In particular the function $\langle f, E_t\rangle/\lambda_t$ is holomorphic, too; this follows from what we just said if the quadratic function $t \mapsto \lambda_t$ has distinct zeroes, and in the case when it has a double zero t_0 we recall that the derivatives $\frac{dE_t}{dt}|_{t=t_0}$ also belong to $\mathrm{Eis}(\lambda)$. Therefore, the integrand in (41) is at least locally integrable in t. To examine absolute convergence of the integral, one reasons exactly as for cusp forms, but rather than pointwise estimates in t one only looks at averages over $T \le t \le T+1$. In place of the L^2-normalization of φ_i we have the following estimate

$$\int_T^{T+1} dt \int_{\mathrm{ht}\le Y} |E_t^j(g)|^2 \ll T^2 + \log(Y)$$

where $\mathrm{ht} \le Y$ means that we integrate over the complement of the set $y \ge Y$ in each cusp. This bound is derived from the Maass–Selberg relations (cf. [Iwa95, Proposition 6.8 and (6.35) and (10.9)]) and average bounds on the scattering matrix (equation (10.13), *op. cit.*). From this, one obtains in the same way as the cuspidal case bounds on $\int_T^{T+1} |\langle f, E_t^j\rangle|^2$ and $\int_T^{T+1} |XE_t^j|^2$ that are of the same quality as (42) and (43) and the same analysis as for the cuspidal spectrum goes through. □

5.6. **Proof of the proposition.** We now prove Proposition 4.1. Write $f = f_s + \sum f_{c_i}$ as in Lemma 5.1. By Lemma 5.2 and Proposition 5.3 there are functions $g_i, g \in \mathcal{A}_\zeta$ with

$$(\mathcal{C}-\lambda)g_i = f_{c_i}, \quad (\mathcal{C}-\lambda)g = f_s,$$

where, in the case of g_i, we use Lemma 5.2 to produce a function on $\widetilde{\mathcal{C}_i}$, and then extend it by zero to get an element of $\mathcal{A}_\zeta$. Then $g + \sum_i g_i$ is the desired preimage of f under $\mathcal{C} - \lambda$. □

6. Interpolation and cohomology.

We will recall background on the Segal-Shale-Weil representation (see [LV13] for details) necessary to explain why the foregoing results imply the interpolation formula of Radchenko and Viazovska. We have already outlined the argument in §1.1 and what remains is to explain in detail where (4) comes from.

6.1. **The Weil representation.** Let $L^2(\mathbf{R})_+$ be the Hilbert space of even square integrable functions on $\mathbf{R}$, and let $\mathcal{S}$ be the subspace of even Schwartz functions, i.e. even smooth functions f such that

$$\sup_{x\in\mathbf{R}} \left| x^n \frac{d^m}{dx^m} f(x) \right| < \infty \tag{45}$$

for any pair (m,n) of non-negative integers. Let G be the degree 2 cover of $SL_2(\mathbf{R})$. There is a unique unitary representation of G on $L^2(\mathbf{R})$, the Weil (or oscillator) representation, for which $\mathcal{S}$ is precisely the subspace of smooth vectors and such that the action of $\mathfrak{g}$ on $\mathcal{S}$ is given by:

$$X\cdot\phi(x) = -i\pi x^2\phi(x), \qquad Y\cdot\phi(x) = \frac{-i}{4\pi}\frac{\partial^2}{\partial x^2}\phi(x), \qquad H\cdot\phi(x) = \left(x\frac{d}{dx} + \frac{1}{2}\right)\phi(x).$$

It then follows that $\kappa = i(X - Y)$ acts by

$$\kappa\cdot\phi(x) = \left(\pi x^2 - \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\right)\phi(x)$$

The normalization ensures that the action of G is unitary and that the relation $\sigma X\sigma^{-1} = Y$ is preserved, where $\sigma : \mathcal{S} \to \mathcal{S}$ is the Fourier transform:

$$\sigma(\phi)(\xi) = \hat{\phi}(\xi) := \int_{\mathbf{R}} \phi(x)e^{-2\pi i x\xi}\,dx.$$

Moreover, with respect to the seminorms of (45), the topological vector space $\mathcal{S}$ has the structure of a moderate growth Fréchet representation of G.

The vector $v_{1/2} := e^{-\pi x^2}$ has κ-weight 1/2 and Casimir eigenvalue $-3/8$. The other K-finite vectors in $\mathcal{S}$ are spanned by its Lie algebra translates; they have the form $q(x)e^{-\pi x^2}$ for q an even polynomial, and have weights $\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \cdots$.

6.2. **The lattice** Γ**.** If $x \in \mathfrak{g}$ is nilpotent, the projection map identifies $\exp(\mathbf{R}x) \subset G$ with the corresponding 1-parameter subgroup of $SL_2(\mathbf{R})$. In particular, the map $G \to SL_2(\mathbf{R})$ splits over any one-parameter unipotent subgroup; thus the groups of upper and lower-triangular matrices have distinguished lifts in G.

In particular, the elements $e = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ and $f = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$ defined in (3) have distinguished lifts $\tilde{e}$, $\tilde{f}$ to G. They act in the Weil representation by:

$$\tilde{e}\cdot\phi(x) = e^{-2\pi i x^2}\phi(x), \qquad \tilde{f}\cdot\phi(x) = \sigma\tilde{e}\sigma^{-1}\phi(x). \tag{46}$$

Let $\Gamma \in SL_2(\mathbf{Z})$ be the subgroup freely generated by e and f. It is the subgroup of $\Gamma(2)$ whose diagonal entries are congruent to 1 mod 4, and is conjugate to $\Gamma_1(4)$.

Lemma 6.1. *There is a splitting $\Gamma \to G$ which extends the splitting over the two subgroups $\langle e \rangle$ and $\langle f \rangle$. The image of Γ in this splitting are precisely the elements of its preimage leaving fixed the distribution $\mathcal{Q} := \sum_{n\in\mathbf{Z}} \delta_{n^2}$ – see 1.1 for the definition of δ_n.*

Proof. The lift $\tilde{e}$ of e to G fixes $\mathcal{Q}$. By Poisson summation, so does the lift $\tilde{f}$ of f. The group $\tilde{\Gamma}$ generated by $\tilde{e}$ and $\tilde{f}$ surjects onto Γ with kernel of size at most two. But $\tilde{\Gamma}$ fixes $\mathcal{Q}$, and the two lifts of any $g \in SL_2(\mathbf{R})$ to G act on $\mathcal{S}$ by different signs, so the map $\tilde{\Gamma} \to \Gamma$ is injective. □

6.3. **Conclusion of the proof.** We now fill in the deduction, already sketched in the introduction, of the Interpolation Theorem 1.1 from Theorem 1.2.

We first handle a detail of topology from the discussion of §1.1, namely, the equivalence between the interpolation statement and its "dual" form. For a Fréchet space F we denote its continuous dual by F^*; we regard it as an abstract vector space without topology. Then, for $\eta : E \to F$ a continuous map of Fréchet spaces,

$$\text{if } \eta^* : F^* \to E^* \text{ is bijective, then } \eta \text{ is a homeomorphism.} \tag{47}$$

Indeed, following [Trè06, Theorem 37.2], a continuous homomorphism $\eta : E \to F$ of Fréchet spaces is surjective if η^* is injective and its image is weakly closed. Applying this in the situation of (47), we see at least that η is surjective. It is injective because the image of η^* is orthogonal to the kernel of η, and then we apply the open mapping theorem to see that it is a homeomorphism. In §1.1, we apply the statement (47) to the map Ψ of Theorem 1.1, with codomain the closed subspace of $\mathfrak{s} \oplus \mathfrak{s}$ defined by $\sum_{n\in\mathbf{Z}} \phi(n) = \sum_{n\in\mathbf{Z}} \widehat{\phi}(n)$.

The other point that was not proved in §1.1 was (4), the actual evaluation of H^0 and H^1 for the dual of the oscillator representation, namely

$$\dim H^0(\Gamma, \mathcal{S}^*) = 1, \quad \dim H^1(\Gamma, \mathcal{S}^*) = 0. \tag{48}$$

Now, $\mathcal{S}^*$ is precisely the distribution globalization of the dual of $\mathcal{S}_K$, i.e. it is the $W_{-\infty}^*$ of the statement of Theorem 1.2 if we take W to be $\mathcal{S}_K$. Therefore Theorem 1.2 reduces us to showing that the multiplicity of W (respectively W^{cl}) in the space of automorphic forms (respectively cusp forms) for Γ equals 1 (respectively 0).

From 6.1, the K-finite vectors $\mathcal{S}_K$ are a realization of the $(\mathfrak{g}, K)$-module of lowest weight 1/2, whose complementary representation $(\mathcal{S}_K)^{cl}$ is the $(\mathfrak{g}, K)$-module of highest weight $-3/2$. In general, a homomorphism from a lowest weight $(\mathfrak{g}, K)$-module to any $(\mathfrak{g}, K)$-module W is uniquely specified by the image of the lowest weight vector, which can be an arbitrary element of W killed by m; and the dual statement about highest weight modules is also valid.

It follows that $(\mathfrak{g}, K)$-homomorphisms from $\mathcal{S}_K$ (respectively $\mathcal{S}_K^{cl}$) to the space $\mathcal{A}$ of automorphic forms correspond exactly to holomorphic forms of weight 1/2 (respectively, antiholomorphic forms of weight $-3/2$); the conditions of being killed by m or p precisely translate to being holomorphic or antiholomorphic. The desired conclusion (48) now follows from:

Lemma 6.2. (a) *The space of holomorphic forms for Γ of weight 1/2 is one-dimensional, and the space of cuspidal holomorphic forms of this weight is trivial.*
(b) *The space of* cuspidal *holomorphic forms for Γ of weight 3/2 is trivial; therefore, the space of cuspidal antiholomorphic forms for Γ of weight $-3/2$ is also trivial.*

Proof. For (a), the group Γ is conjugate to $\Gamma_1(4)$, for which the space of modular forms of weight 1/2 is spanned by the theta series $\theta_{1/2}(z) = \sum_{n\in\mathbf{Z}} e^{2\pi i z n^2}$ [SS77].

For (b), we use the fact that multiplication by θ injects the space of weight 3/2 forms into the space of weight 2 forms. The space of weight 2 cusp forms for $\Gamma_1(4)$ is, however, trivial; indeed, the compactified modular curve $X_1(4)$ has genus zero. The final assertion follows by complex conjugation. $\square$

6.4. **Variants: odd Schwartz functions, higher dimensions, $\mathbf{P}_{\mathbf{R}^1}$.** We now show how the same ideas give several other interpolation theorems without changing the group $\Gamma = \langle e, f\rangle$; it may also be of interest to consider (∞, p, q)-triangle groups.

6.4.1. *Odd Schwartz functions.* The discussion of Section 6.1 on the even Weil representation $\mathcal{S}$ carries verbatim to its odd counterpart $\mathcal{T}$, whose $(\mathfrak{g}, K)$-module of K-finite vectors is spanned by the translates of the lowest weight vector $v_{3/2} = xe^{-\pi x^2}$. As above, we compute using Theorem 1.2, to get

$$H^0(\Gamma, \mathcal{T}^*) = \mathbf{C}, \quad H^1(\Gamma, \mathcal{T}^*) = 0.$$

Indeed, the zeroth cohomology $H^0(\Gamma, \mathcal{T}^*)$ is identified with the space of modular forms of weight 3/2, a one-dimensional space spanned by θ^3. As for $H^1(\Gamma, \mathcal{T}^*)$, its dimension is equal to the multiplicity of $\mathcal{T}^{cl}$ in the space of cusp forms on Γ. The representation $\mathcal{T}^{cl}$ has highest weight $-1/2$, and the vanishing of H^1 results from the absence of cusp forms of weight 1/2 on Γ as in 6.2. We then deduce an interpolation theorem as in §1, noting that in addition to the δ_n the distributions $\phi \mapsto \phi'(0)$ (resp. $\phi \mapsto \hat{\phi}'(0)$) are also e- (resp. f-)invariant. Arguing as in §1.1 recovers a non-explicit version of the interpolation theorem of Radchenko-Viazovska for odd Schwartz functions, see [RV19, Theorem 7].

6.4.2. *Schwartz functions on* $\mathbf{R}^d$. We may, similarly, consider instead the representation $\mathcal{S}_d$ of $\mathrm{SL}_2(\mathbf{R})$ on radial Schwartz functions on $\mathbf{R}^d$. This is, for reasons very similar to that enunciated in §6.1, a lowest weight representation of the double cover of $\mathrm{SL}_2(\mathbf{R})$, but now of lowest weight $d/2$ generated by $e^{-\pi(x_1^2+\ldots+x_d^2)}$. We claim that in all cases the corresponding H^1 continues to vanish. Indeed, for d even the complementary representation W^{cl} is finite-dimensional and does not occur in cusp forms; for d odd, occurrences of W^{cl} in cusp forms correspond just as before to holomorphic cusp forms of weight $\frac{4-d}{2}$ for $\Gamma(2)$, and these do not exist for any odd d. Therefore we find that the values of f and $\hat{f}$ at radii $\sqrt{n}$ determine f, subject only to a finite-dimensional space of constraints, of dimensional equal to the dimension of weight $d/2$ holomorphic forms for $\Gamma(2)$.

6.4.3. *Smooth functions on the projective real line.* We consider the space $C^\infty(\mathbf{P}^1_{\mathbf{R}})$ of smooth functions on $\mathbf{P}^1_{\mathbf{R}} = \mathbf{R} \cup \infty$, which we may think of equivalently as either $\mathbf{R}^\times$-invariant smooth functions $\Phi(x, y)$ on $\mathbf{R}^2 - \{0\}$ or smooth functions $h(x)$ on $\mathbf{R}$ with an asymptotic expansion $h \sim c_0 + c_1/x + \ldots$ as $x \to \infty$; to pass from the former to the latter we use $h(x) = \Phi(x, 1)$. We define for $\Phi \in C^\infty(\mathbf{P}^1_{\mathbf{R}})$:

$$a_n(\Phi) = \int \Phi(x, 1) e^{\pi i n x} dx \quad \text{and} \quad b_n(\Phi) = \int \Phi(1, y) e^{\pi i n y} dy,$$

These integrals are not convergent, but are defined by regularization. For example, $a_n(\Phi)$ should be understood as the convergent limit of $\int_{|x| \leq T} (\Phi(x, 1) - \Phi(1, 0)) e^{\pi i n x} dx$ as $T \to \infty$. In terms of h, they are the (regularized) Fourier transforms of $h(x)$ and

$h(1/x)$ at half-integers. Integration by parts shows that a_n, b_n decay faster than any polynomial; in other words, they belong to the space $\mathfrak{s}$ introduced in §1.

Theorem 6.3. *The map $h \mapsto (a_n, b_n, h(0), h'(0), h(\infty), h'(\infty))$ defines topological isomorphisms of $C^\infty(\mathbf{P}^1_{\mathbf{R}})$ with a codimension 5 subspace of $\mathfrak{s}^2 \oplus \mathbf{C}^4$.*

In principle, the 5 constraints can be explicated by means of the study of the boundary values of weight 2 Eisenstein series, but the details are a little delicate, and we did not carry out the computation.

Proof. We obtain Theorem 6.3 in a similar way to Theorem 1.1 – namely, by applying Theorem 1.2 for the same Γ, but with a different coefficient system. Namely, we consider the space$W = C^\infty(\mathbf{P}^1_{\mathbf{R}})$ and apply Theorem 1.2 to the $(\mathfrak{g}, K)$-module W_K.

Here, the e-invariants on W^* are spanned by spanned by the $\Phi \mapsto a_n(\Phi)$ together with $\Phi(1,0)$ and $\partial_y \Phi(1,0)$ (i.e., $h(\infty)$ and $h'(\infty)$). To check this, take an arbitrary e-invariant distribution $\mathcal{D}$ on W; its restriction to $C_c^\infty(\mathbf{R})$ is a periodic distribution under $x \mapsto x+2$ and is thus readily verified to be a linear combination of the distributions a_n for $n \in \mathbf{Z}$. We may therefore suppose that $\mathcal{D}$ vanishes on $C_c^\infty(\mathbf{R})$, and is so supported at ∞, but it is then a linear combination of the various coefficients c_i in the asymptotic expansion of

$$h(x) \sim c_0 + \frac{c_{-1}}{x} + \frac{c_{-2}}{x^2} + \dots$$

and a computation shows that only c_0, c_{-1} are invariant under the translation map $x \mapsto x+2$. Symmetrically, W^f is spanned by the functionals $b_n(\Phi), h(0), h'(0)$. Therefore, to prove the theorem, it is enough to show that $(W^*)^e$ and $(W^*)^f$ add up to W^*, and intersect in a 5-dimensional space.

The space W is identified with a reducible principal series of $SL_2(\mathbf{R})$, namely with the extension $\mathbf{C} \to W \to D_2^{\pm}$ of the discrete series of weight ± 2 by the trivial representation. Either via Theorem 1.2 (plus an auxiliary computation) or directly from the results of [BO98, §8] one computes that

$$\dim H^1(\Gamma, W^*) = 2, \qquad H^0(\Gamma, W^*) = 5. \tag{49}$$

in fact the natural map $W^* \to \mathbf{C}$ induces an isomorphism on H^1. We apply again the Mayer-Vietoris sequence (5). It becomes, using subscripts and superscripts to denote coinvariants and invariants respectively,

$$0 \to H^0(\Gamma, W^*) \to (W^*)^e \oplus (W^*)^f \to (W^*) \to H^1(\Gamma, W^*) \to W^*_e \oplus W^*_f \to 0. \tag{50}$$

The constant function in W is e-invariant and f-invariant. By evaluating at it we see that both W^*_e and W^*_f are at least one-dimensional. But then comparing (50) and (49) we find that the surjective map $H^1(\Gamma, W^*) \to W^*_e \oplus W^*_f$ must be an isomorphism, and so $(W^*)^e \oplus (W^*)^f \to (W^*)$ is surjective with 5-dimensional kernel as required. □

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